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$\mathbf{W_\infty}$ and $\mathbf{SL_q(2)}$ Algebras in the Landau Problem and Chern-Simons Theory on a Torus

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Abstract

We discuss w_∞ and $sl_q(2)$ symmetries in Chern-Simons theory and Landau problem on a torus. It is shown that when the coefficient of the Chern-Simons term, or when the total flux passing through the torus is a rational number, there exist in general two w_∞ and two $sl_q(2)$ algebras, instead of one set each discussed in the literature. The general wavefunctions for the Landau problem with rational total flux is also presented.

Chern-Simons (CS) field theory with matter coupling have attracted intense interest in recent years, owing to its relevance to condensed matter systems such as quantum Hall systems, and possibly high T_c superconductivity. While the majority of works in the field is concerned with planar systems, Chern-Simons field theory on compact Riemann surfaces has also captured considerable interests. It has even richer structures, which, being topological in nature, are absent in planar system. Among them are the multicomponent structure of many-body wavefunctions and the degeneracy of physical states (see references in [1]).

Many studies have also been carried out for the Maxwell-Chern-Simons (MCS) theory in which a kinetic term for the CS gauge field is included. An interesting observation is that the dynamics in the topological sector of MCS theory on a torus is equivalent to the Landau problem (*i.e.* charged particle moving in a constant magnetic field) on a torus [2-4]. Thus many interesting features are shared by both systems.

Recently a w_∞ symmetry is uncovered in the Landau problem and in the related problem of fractional quantum Hall effects [5-7]. More recently, a $sl_q(2)$ quantum algebra is realized in these systems [8,6*b*,9]. Representation of this quantum algebra was applied to formulate the Bethe-*ansatz* for the problem of Bloch electron in magnetic field, *i.e.* the Azbel-Hofstadter problem [8]. Naturally, these symmetries were realized also in the MCS and the pure CS theory on a torus [6*b*].

The w_∞ and $sl_q(2)$ algebras considered in these works are mainly for the situations in which the total flux passing through the torus (or unit cell in the lattice) is an integral multiple of 2π, or equivalently in the CS theory when the CS coefficient is an integer. In this paper, we would like to point out that in the more general case, that is, when the flux or the CS coefficient is rational, there could exist two w_∞ and two $sl_q(2)$ algebras.

Let us consider a Maxwell-Chern-Simons (MCS) theory defined on a torus with lengths L_1 and L_2. The Lagrangian is given by

$$L = -\frac{\alpha}{4} f_{\mu\nu}^2 + \frac{\kappa}{4\pi}\, \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \ . \tag{1}$$

Here we assume κ to be a rational number, $\kappa = N/M$, where N, M are two coprime integers. The equations of motion can be solved in the $a_0 = 0$ gauge [3,4]. The spatial components a_j are found to decompose into global excitations, $\theta_i(t)/L_i$, and local excitations. Contributions of these two parts decouple in the action of the theory. The global excitations θ_i are the non-integrable phases associated with the two non-contractible loops of the torus. They are responsible for the topological structures of the theory. In the rest of the paper, we will consider the structures of the Hilbert space of the global excitations.

The Lagrangian of the θ_i's is given by

$$\mathcal{L} = \frac{\alpha}{2}\left(\frac{L_2}{L_1}\dot{\theta}_1^2 + \frac{L_1}{L_2}\dot{\theta}_2^2\right) + \frac{\kappa}{4\pi}\left(\theta_2\dot{\theta}_1 - \theta_1\dot{\theta}_2\right), \tag{2}$$

from which the corresponding canonical momenta are obtained:

$$\begin{aligned} p_1 &= \alpha\frac{L_2}{L_1}\dot{\theta}_1 + \frac{\kappa}{4\pi}\theta_2 \ , \\ p_2 &= \alpha\frac{L_1}{L_2}\dot{\theta}_2 - \frac{\kappa}{4\pi}\theta_1 \ , \\ [\theta_i, p_k] &= i\delta_{ik} \ . \end{aligned} \tag{3}$$

As mentioned previously, the system defined by (2) is equivalent to the Landau problem on a torus. In fact, if one defines $x_i \equiv \theta_i/L_i$, $L_{x_i} \equiv 2\pi/L_i$, $m \equiv \alpha L_1 L_2$ and $B \equiv 2\pi\kappa/L_{x_1}L_{x_2}$, then the Hamiltonian governing the dynamics is

$$H = \frac{1}{2m}\left[\left(p_{x_1} - \frac{1}{2}Bx_2\right)^2 + \left(p_{x_2} + \frac{1}{2}Bx_1\right)^2\right] \ , \tag{4}$$

where $p_{x_i} = m\dot{x}_i + \frac{1}{2}\epsilon^{ik}Bx_k$, $[x_i, p_{x_k}] = i\delta_{ik}$. This is just the Hamiltonian for a charged particle with mass m moving in constant magnetic field B perpendicular to the torus in

the symmetric gauge. The total flux on the torus is $\Phi = BL_{x_1}L_{x_2} = 2\pi\kappa$. The energy spectrum and wavefunctions of this problem has been considered before for the case when $\kappa = N$ $(M = 1)$ [3,10-13]. It is found that the Landau levels are N-fold degenerate. Wen and Niu [3] consider the case $\kappa = 1/M$ in connection with the problem of Fractional Quantum Effect, and show that the degeneracy of states is M. One of the purpose of this paper is to show that in the general case $\kappa = N/M$, the Landau levels are MN-fold degenerate, and to present an explicit expression for these wavefunctions.

To construct the wavefunctions, one has to first identify the appropriate symmetry algebra of the system. The wavefunctions must form a representation of the algebra. For the Landau problem, the relevant algebra is the Weyl-Heisenberg (WH) algebra of the so-called magnetic translation operators (MTO) [14]. What we would like to emphasize here is that in general there are two commuting WH algebras in the system, and not just one as usually considered in the literature. One of these two algebras leads to an N-fold degeneracy, while the other to a M-fold degeneracy, giving rise to a total of MN degenerate states.

The MTO in this problem are given by [14] (no summation over repeated indices)

$$U_j = \exp\left\{ L_{x_j}\left(ip_{x_j} + \frac{i}{2}\epsilon^{jk} B x_k \right) \right\} \ . \tag{5}$$

In the original MCS theory they are the generators of large gauge transformations inducing $\theta_j \rightarrow \theta_j + 2\pi$:

$$U_j = \exp\left(2\pi i p_j + \frac{i}{2}\kappa\epsilon^{jk}\theta_k \right) \ . \tag{6}$$

They commute with the Hamiltonian (4). Now there is another set of operators which commute with H and U_j. These operators are defined by $V_j \equiv U_j^{1/\kappa}$. U_j and V_j satisfy dual WH algebras :

$$\begin{aligned} U_1U_2 &= e^{-2\pi i\kappa}\, U_2U_1 \ , \\ V_1V_2 &= e^{-2\pi i/\kappa}\, V_2V_1 \ , \end{aligned} \tag{7}$$

and $[U_j, V_k] = 0$. The V_j have sometimes been loosely called "magnetic translation operators" in some papers (And this, we believe, does cause some confusion). Now the eigenfunctions ψ of H must form a representation of (7). In previous works on Landau problem on a torus, only one of the two WH algebras in (7) were employed in the study of the structure of Hilbert space. To get a complete Hilbert space when the total flux through the torus is rational, one must consider both algebras. By noting that U_1^M, U_2^M, V_1^N and V_2^N commute with each other, we can choose ψ to be eigenstates of U_i^M and V_j^N : $U_i^M\psi = V_i^N\psi = e^{i\alpha_i}\psi$ $(i = 1, 2)$, where α_i are the vacuum angles. The problem is more easily solved in terms of complex variables. Following the procedure in [11,12], we define $u_a = \sqrt{B}x_a$, $l_a = \sqrt{B}L_{x_a}$ $(l_1 l_1 = 2\pi\kappa)$ and $z = u_1 + iu_2$. Then the Hamiltonian is $H = \frac{B}{m}\left(\bar{c}c + \frac{1}{2}\right)$, where $c \equiv i\sqrt{2}\left(\partial_z + \frac{1}{4}\bar{z}\right)$, $\bar{c} \equiv i\sqrt{2}\left(\partial_{\bar{z}} - \frac{1}{4}z\right)$ and $[c, \bar{c}] = 1$. Hence the energy spectrum is $E = \frac{B}{m}\left(n + \frac{1}{2}\right), n = 0, 1, 2, \ldots$. The wavefunctions of Landau levels are given by $\psi_n(z,\bar{z}) = \bar{c}^n\psi_0(z,\bar{z})$, where the ground state ψ_0 satisfies $c\psi_0 = 0$. The problem of finding the general wavefunctions now reduces to that of finding ψ_0. To this end, we note that the requirements $U_i^M\psi = e^{i\alpha_i}\psi$ stated above imply the following boundary conditions of ψ in complex variable form:

$$
\begin{aligned}
\psi(z + Ml_1, \bar{z} + Ml_1) &= e^{-\frac{1}{4}Ml_1(z-\bar{z})+i\alpha_1}\ \psi(z,\bar{z})\ , \\
\psi(z + iMl_2, \bar{z} - iMl_2) &= e^{+\frac{1}{4}iMl_2(z+\bar{z})+i\alpha_2}\ \psi(z,\bar{z})\ .
\end{aligned}
\tag{8}
$$

We now let ψ_0 assume the form:

$$
\psi_0(z,\bar{z}) = \xi(z,\bar{z})\ G(\bar{z})\ , \tag{9}
$$

where $G(\bar{z})$ is a function of $\bar{z}$ only. Then the condition $c\psi_0 = 0$ implies $c\xi = 0$, which is solved by

$$
\xi = \exp\left(-\frac{1}{4}z\bar{z} + \frac{1}{4}\bar{z}^2 + i\delta_1\bar{z}\right)\ \ ,\ \delta_j \equiv \alpha_j/Ml_j\ . \tag{10}
$$

From (9) and (10) one finds that the function G obeys the following boundary conditions

$$
\begin{aligned}
G(\bar{z} + Ml_1) &= G(\bar{z}) \ , \\
G(\bar{z} - iMl_2) &= e^{\frac{1}{2}(Ml_2)^2 + iMl_2(\bar{z}+i\delta)} \ G(\bar{z}) \ ,
\end{aligned}
\tag{11}
$$

where $\delta = \delta_1 - i\delta_2$. Hence G is just a doubly periodic function related to the theta functions. Since the area bounded by the two translations Ml_1 and Ml_2 on the z plane is $M^2 l_1 l_2 = 2\pi MN$ (recall $l_1 l_2 = 2\pi\kappa$), there exist MN independent solutions of G. These functions are conveniently labeled by two indices j, k ($j = 0, \ldots, N-1;\ k = 0, \ldots, M-1$) and are given by

$$
G_{jk}(\bar{z}) = \sum_{n=-\infty}^{\infty} Q^{\frac{(MNn - Nk - Mj)^2}{MN}} e^{-2\pi i (MNn - Nk - Mj)(\bar{z}+i\delta)/Ml_1} \ . \tag{12}
$$

Here $Q \equiv e^{-\pi l_2/l_1}$. So each Landau level is MN-fold degenerate : $\psi_{njk} = \bar{c}^n \psi_{ojk}$ and $\psi_{0jk}(z,\bar{z}) = \xi(z,\bar{z}) \ G_{jk}(\bar{z})$. The function $G_{jk}(\bar{z})$ is expressible in terms of the theta functions with characteristics:

$$
\begin{aligned}
G_{jk}(z,\bar{z}) &= \vartheta \begin{bmatrix} -\frac{k}{M} - \frac{j}{N} \\ -i\frac{N}{l_1}\delta \end{bmatrix} \left(-\frac{N}{l_1}\bar{z} \ , \ iMN\frac{l_2}{l_1} \right) \ , \\
\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z,\tau) &= \sum_n \exp\Big(i\pi \left(n+a\right)^2 \tau + 2\pi i \left(n+a\right)\left(z+b\right) \Big) \ .
\end{aligned}
\tag{13}
$$

.

We now consider the actions of the $U_j,\ V_j$ on ψ_{njk}. To simplify discussions, we will concentrate only on the first Landau level $n = 0$. The discussion given below can be carried over directly to higher levels, since the U_j and V_j commute with c and $\bar{c}$. So let us denote by $\langle z, \bar{z} | jk \rangle \equiv \psi_{0jk}(z,\bar{z})$. The actions of $U_i,\ V_i$ on the state vectors are :

$$
\begin{aligned}
U_1 |jk\rangle &= e^{i(\alpha_1 + 2\pi Nk)/M} |jk\rangle \quad , \\
U_2 |jk\rangle &= e^{i\alpha_2/M} |j, k-1\rangle \quad , \\
V_1 |jk\rangle &= e^{i(\alpha_1 + 2\pi Mj)/N} |jk\rangle \quad , \\
V_2 |jk\rangle &= e^{i\alpha_2/N} |j-1, k\rangle \quad .
\end{aligned}
\tag{14}
$$

In the limit $\alpha \to 0$, the MCS theory reduces to the pure CS theory. This corresponds to the reduction of the Hilbert space of the Landau problem to the first level. The Lagrangian now becomes $\mathcal{L} = \frac{\kappa}{4\pi}\left(\theta_2\dot{\theta}_1 - \theta_1\dot{\theta}_2\right)$ and H vanishes identically. From the Lagrangian one sees that θ_1 and θ_2 become canonical variables of the reduced phase space , and thus $[\theta_1,\ \theta_2] = 2\pi i/\kappa$. The Hilbert space structure of this theory is well studied [1-4,15-17]. The operators U_j now reduce to the form $U_j = \exp\left\{i\epsilon^{jk}\kappa\,\theta_k\right\}$, which is still the generators of large gauge transfomations in the theory. For the V_j, one has $(V_1, V_2) \to (e^{i\theta_2}, e^{-i\theta_1}) \equiv (W_2, W_1^{-1})$. The operators W_j are nothing but the Wilson line operators. It is easy to check that W_j satisfy the same WH algebra as the V_j: $W_1W_2 = e^{-2\pi i/\kappa}\,W_2W_1$ and $[W_k, U_j] = 0$. As before, the wavefunctions must form a representation of the two WH algebras. To comform to the expressions given in previous work [1], we shall work in the θ_1-representation and diagonalize both U_1 and W_1 . The resulted states u_{jk} are NM-degenerate : $u_{jk}(\theta_1) = \langle\theta_1|jk\rangle = e^{ik(\alpha_2+N\theta_1)/M+i\alpha_1\theta_1/2\pi M}\ \delta_{2\pi}[\theta_1 + (\alpha_2 - 2\pi Mj)/N]$. Actions of the U_i, W_j on $|jk\rangle$ are easily found to be given by (14), except that in the third and fourth expresions, the V_j are replaced by W_j, and (α_1, α_2) are replaced by $(-\alpha_2, \alpha_1)$ (recall that $V_1 \to W_2$ and $V_2 \to W_1^{-1}$) [1].

We now turn to discuss the quantum w_∞-symmetries and quantum algebras hidden in the MCS and the Landau problem. It is now well known that the magnetic translation operators, or to be more precise, the generators of the WH algebra, span the quantum w_∞ algebra (also termed FFZ algebra [18]) [5,6,9a], and can induce a quantum symmetry $sl_q(2)$, where q is the deformed parameter [8,6,9]. However, only one set of WH algebra was considered in the works just cited. Particularly, in [6] and [9b] only the case for which the total flux through the torus is $\Phi = 2\pi N$ $(M = 1)$ is considered. As we see before, there are in general two WH algebras present when $\Phi = 2\pi N/M$ and $N \neq M \neq 1$. One thus expect to find two quantum w_∞ algebras and two $sl_q(2)$. This is indeed the case. Below

we construct the generators for these algebras in the MCS theory (which is the same as the Landau problem). The entire results can be easily extended to the pure CS theory by making the appropriate changes in V_j, W_j and α_j mentioned previously.

Let us define the following operators [5,19] :

$$\begin{aligned} S_{\mathbf{n}} &= S_{(n_1,n_2)} \equiv q^{n_1 n_2/2} U_1^{n_1} U_2^{n_2} \ , \\ T_{\mathbf{n}} &= T_{(n_1,n_2)} \equiv \bar{q}^{n_1 n_2/2} V_1^{n_1} V_2^{n_2} \ , \end{aligned} \tag{15}$$

where n_1, n_2 are integers , $q \equiv e^{2\pi i\kappa}$ ($q^M = 1$) and $\bar{q} \equiv e^{2\pi i/\kappa}$ ($\bar{q}^N = 1$). $S_{\mathbf{n}}$ and $T_{\mathbf{n}}$ are simply operators that generate general magnetic translations on the torus. Using the WH algebras (7) one obtains :

$$\begin{aligned} S_{\mathbf{m}}\ S_{\mathbf{n}} &= q^{-\mathbf{m}\times\mathbf{n}/2} S_{\mathbf{m}+\mathbf{n}} \ , \\ T_{\mathbf{m}}\ T_{\mathbf{n}} &= \bar{q}^{-\mathbf{m}\times\mathbf{n}/2} T_{\mathbf{m}+\mathbf{n}} \ . \end{aligned} \tag{16}$$

Here $\mathbf{m}\times\mathbf{n} = m_1 n_2 - m_2 n_1$. From (16) one easily gets the two quntum w_∞ mentioned before:

$$\begin{aligned} [S_{\mathbf{m}}, S_{\mathbf{n}}] &= -2i \sin\Big(i\pi\kappa\,(\mathbf{m}\times\mathbf{n})\Big) S_{\mathbf{m}+\mathbf{n}} \ , \\ [T_{\mathbf{m}}, T_{\mathbf{n}}] &= -2i \sin\left(\frac{i\pi}{\kappa}\,(\mathbf{m}\times\mathbf{n})\right) T_{\mathbf{m}+\mathbf{n}} \ , \end{aligned} \tag{17}$$

Only the T-algbra was given in [6]. Using (14), the actions of $S_{\mathbf{n}}$ and $T_{\mathbf{n}}$ on the state $|jk\rangle$ are found to be:

$$\begin{aligned} S_{\mathbf{n}}|jk\rangle &= q^{\frac{n_1 n_2}{2}}\, e^{i\alpha_2 \frac{n_2}{M}}\, e^{in_1[\alpha_1 + 2\pi(k-n_2)N]/M} |j, k-n_2\rangle \\ T_{\mathbf{n}}|jk\rangle &= \bar{q}^{\frac{n_1 n_2}{2}}\, e^{i\alpha_2 \frac{n_2}{N}}\, e^{in_1[\alpha_1 + 2\pi(j-n_2)M]/N} |j-n_2, k\rangle \ . \end{aligned} \tag{18}$$

When $\alpha_i = 0$ and $M = 1$, the expression for $T_{\mathbf{n}}$ reduces to eq.(3.19) in [6*b*] (note that $\kappa/2$ in [6*b*] is equivalent to κ here).

From the operators (15) one can construct two quantum algebras $sl_q(2)$ and $sl_{\bar{q}}(2)$. The generators of a general $sl_q(2)$ are defined by:

$$\begin{aligned} q^{J_3} J_\pm q^{-J_3} &= q^{\pm 1} J_\pm \ , \\ [J_+, J_-] &= [2J_3]_q \ , \end{aligned} \tag{19}$$

where $[x] \equiv (q^x - q^{-x})/(q - q^{-1})$. In our case, the J's for the $sl_q(2)$ are constructed from the $S_{\mathbf{n}}$ as follows [6,9]:

$$
\begin{aligned}
J_+ &\equiv \frac{1}{q - q^{-1}} \left(S_{(1,1)} - S_{(-1,1)} \right) \ , \\
J_- &\equiv \frac{1}{q - q^{-1}} \left(S_{(-1,-1)} - S_{(1,-1)} \right) \ , \\
q^{2J_3} &\equiv S_{(-2,0)} \ , \qquad\qquad q^{-2J_3} \equiv S_{(2,0)} \ .
\end{aligned}
\tag{20}
$$

The generators $\bar{J}$'s for the $sl_{\bar{q}}(2)$ algebra are also constructed in the same way as in (20), with J, q, S replaced by $\bar{J}, \bar{q}, T$ respectively.

With the help of (18), it is easy to obtain the actions of the $J, \bar{J}$ on the state vectors $|jk\rangle$:

$$
\begin{aligned}
J_+ |jk\rangle &= e^{i\alpha_2/M} \left[k - \frac{1}{2} + \frac{\alpha_1}{2\pi N} \right]_q |j, k-1\rangle \ , \\
J_- |jk\rangle &= -e^{-i\alpha_2/M} \left[k + \frac{1}{2} + \frac{\alpha_1}{2\pi N} \right]_q |j, k+1\rangle \ , \\
q^{\pm 2J_3} |jk\rangle &= q^{\mp 2\left(k + \frac{\alpha_1}{2\pi N}\right)} |jk\rangle \ ,
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
\bar{J}_+ |jk\rangle &= e^{i\alpha_2/N} \left[j - \frac{1}{2} + \frac{\alpha_1}{2\pi M} \right]_{\bar{q}} |j-1, k\rangle \ , \\
\bar{J}_- |jk\rangle &= -e^{-i\alpha_2/N} \left[j + \frac{1}{2} + \frac{\alpha_1}{2\pi M} \right]_{\bar{q}} |j+1, k\rangle \ , \\
\bar{q}^{\pm 2\bar{J}_3} |jk\rangle &= \bar{q}^{\mp 2\left(j + \frac{\alpha_1}{2\pi M}\right)} |jk\rangle \ ,
\end{aligned}
\tag{22}
$$

In view of (14) the state functions $|jk\rangle$ in general form a cyclic representation of $sl_q(2) \times sl_{\bar{q}}(2)$ with dimension $M \times N$. However, under certain choices of the boundary conditions α_i, highest weight representation can be formed [20]. For simplicity, let us first give a general consideration for the subspace $|k\rangle = |j, k\rangle$ acted by the $sl_q(2)$ algebra ($q^M = 1$). Now $|k + M\rangle = |k\rangle$ implies that the states form a M-dimensional cyclic representation of $sl_q(2)$ in general. But from (21), and the observation that $[\frac{M}{2}]_q = 0$, one concludes that if the boundary condition $\lambda_N \equiv \alpha_1/2\pi N$ is such that $l = \frac{M}{2} - \left(\lambda_N + \frac{1}{2}\right)$ is

an integer, then the state $|l+1\rangle$ is the highest and $|l\rangle$ is the lowest weight states:

$$
\begin{aligned}
J_+|l+1\rangle &\sim \left[\frac{M}{2}\right]_q |l\rangle = 0 \ , \\
J_-|l\rangle &\sim -\left[\frac{M}{2}\right]_q |l+1\rangle = 0 \ .
\end{aligned}
\tag{23}
$$

The succesive actions of J_- starting from the highest weight state is: $|l+1\rangle \to \cdots \to |M\rangle = |0\rangle \to |1\rangle \to \cdots \to |l\rangle$ (J_+ acts reversely). The condition that l be an integer requires that $\lambda_N = m$ for odd M, and $\lambda_N + \frac{1}{2} = m$ for even M (m =integers). Similar consideration leads to the same requirements for $\lambda_M \equiv \alpha_1/2\pi M$ for the $sl_{\bar{q}}(2)$ algebra : $\lambda_M(\lambda_M + \frac{1}{2})$ = integers for odd (even) N. Combining these considerations for both sectors, one concludes that the states $|j,k\rangle$ can form highest weight representation for both $sl_q(2)$ and $sl_{\bar{q}}(2)$ algebras only when M and N are both odd, and $\alpha_1 = n \times MN$ where n are integers. There is no restriction on the α_2. This unbalanced requirement in the two vacuum angles is only due to our diagonalizing both U_1 and V_1. The situation changes if we diagonalize instead U_1 and $W_1 = V_2^{-1}$ as in the pure CS theory, then both vacuum angles are constrained in order to have highest weight representation for both algebras. In this case, $\alpha_1/2\pi N$ ($\alpha_2/2\pi M$) $= m$ or $m - \frac{1}{2}$ if M (N) is odd or even respectively.

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References

[1] C.-L. Ho and Y. Hosotani, *Int. J. Mod. Phys.* **A7**, 5797 (1992); *Phys. Rev. Lett.* **70**, 1360 (1993); D. Wesolowski, Y. Hosotani and C.-L. Ho, *Int. J. Mod. Phys.* **A9**, 969 (1994).

[2] I.I.Kogan and A. Yu. Morozov, *Sov. Phys. JETP* **61**, 1 (1985); I.I. Kogan, ITEP preprint ITEP-89-163 (1989); *Comm. Nucl. Part. Phys.* **19**, 305 (1990).

[3] X.G. Wen, *Phys. Rev.* **B40**, 7387 (1989); *Int. J. Mod. Phys.* **B4**, 239 (1990); X.G. Wen and Q. Niu, *Phys. Rev.* **B41**, 9377 (1990).

[4] A.P. Polychronakos, *Ann. Phys. (N.Y.)* **203B**, 231 (1990); Univ. of Florida preprint, UFIFT-HEP-89-9; *Phys. Lett.* **241B**, 37 (1990).

[5] T. Dereli and A. Vercin, *Phys. Lett.* **288B**, 109 (1992); *J. Phys.* **A26**, 6961 (1993).

[6] I.I. Kogan, a)*Mod. Phys. Lett.* **A7**, 3717 (1992); b) *Int. J. Mod. Phys.* **A9**, 3889 (1994).

[7] S. Iso, D. Karabali and B. Sakita, *Phys. Lett.* **296B**, 143 (1992); B. Sakita, *Phys. Lett.* **315B**, 124 (1993); A. Cappelli, C.A. Trugenberger and G.R. Zembra, *Nucl. Phys.* **B396**, 465 (1993); A. Cappelli, G.V. Dunne, C.A. Trugenberger and G.R. Zembra, *Nucl. Phys.* **B398**, 531 (1993).

[8] P.B. Wiegmann and A.V. Zabrodin, cond-mat/9310017; cond-mat/9312088; L.D. Faddeev and R.M. Kashaev, hep-th/9312133.

[9] H.-T. Sato,a) *Mod. Phys. Lett.* **A9**, 451 (1994); b) *ibid.* 1819; *Prog. Theor. Phys.* **93**, 195 (1994).

[10] R.B. Laughlin, *Ann. Phys. (N.Y.)* **191B**, 163 (1989).

[11] C.-L. Ho, *Mod. Phys. Lett.* **A5**, 2029 (1990).

[12] M. Burgess, A. McLachlan and D.J. Toms, *Int. J. Mod. Phys.* **A8**, 2623 (1993).

[13] For Landau problem in higher genus Riemann surfaces, see: R. Iengo and D. Li, *Nucl. Phys.* **B413**, 735 (1994).

[14] E. Brown, *Phys. Rev.* **133**, 1038 (1963); J. Zak, *Phys. Rev.* **134**, 1602 (1964); *ibid.* 1607.

[15] Y. Hosotani, *Phys. Rev. Lett.* **62**, 2785 (1989); *Phys. Rev. Lett.* **64**, 1691 (1990).

[16] K. Lee, Boston Univ. report, BU/HEP-89-28;

[17] R. Iengo and K. Lechner, *Nucl. Phys.* **B346**, 551 (1990); *Nucl. Phys.* **B364**, 551 (1991); K. Lechner, *Phys. Lett.* **273B**, 463 (1991); Trieste SISSA preprint, Thesis, Apr 1991.

[18] D.B. Fairlie, P. Fletcher and C.K. Zachos, *Phys. Lett.* **218B**, 203 (1989); *J. Math. Phys.* **31**, 1088 (1990).

[19] M. Golenishcheva-Kutuzova and D. Lebedev, *Comm. Math. Phys.* **148**, 403 (1992); J. Hoppe, M. Olshanetsky and S. Theisen, *Comm. Math. Phys.* **155**, 429 (1993).

[20] For an extensive discussion on the rôles of the cyclic and highest weight representations of quantum algebras in physics, particularly in the theory of quantum spin chains and the conformal field theories, please see : V. Pasquier and H. Saleur, *Nucl. Phys.* **B330**, 523 (1990); Z.-Q. Ma, *Yang-Baxter Equation and Quantum Enveloping Algebras*, World Scientific, Singapore, 1993.